THE A-POLYNOMIAL FROM THE NONCOMMUTATIVE VIEWPOINT

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1. Introduction

This paper places the A-polynomial of a knot into the framework of noncommutative geometry. The A-polynomial was introduced in [CCGLS]. The A-polynomial describes how the $Sl_2\mathbb{C}$-characters of a knot lie inside the $Sl_2\mathbb{C}$-characters of its boundary torus. It is known that the A-polynomial can be related to the Alexander polynomial of the knot and to the structure of essential surfaces in the complement of the knot [CL].

We construct an invariant depending on a complex parameter, by following the construction of the A-polynomial, but replacing $Sl_2\mathbb{C}$-characters of surfaces and 3-manifolds by the Kauffman bracket skein module of cylinders over surfaces and 3-manifolds. The Kauffman bracket skein module depends on a parameter t, so that when t is set equal to -1 the $Sl_2\mathbb{C}$-characters are recovered. The A-polynomial can be derived from the noncommutative invariant when t is set equal to -1.

The construction of the noncommutative invariant uses the relationship between the Kauffman bracket skein module and the noncommutative torus, already explicated in [FG]. The noncommutative torus is one of the fundamental examples of a noncommutative space [Co].

The invariant is a minimal reduced Gröbner basis of a left ideal in the noncommutative plane. The use of Gröbner bases is an important theme in modern commutative algebra. For instance, algorithms associated with Gröbner bases are the heart of most symbolic manipulation programs. The use of Gröbner bases to study the noncommutative plane is foreshadowed by the work of Cohn on skew polynomial rings [C].

There is a left action of the Kauffman bracket skein module of a cylinder over a torus on the Kauffman bracket skein module of a solid torus. This allows us to compare the noncommutative invariant with

data coming from colored Jones polynomials. Specifically, the noncommutative invariant annihilates the data from the colored Jones polynomials. This should lead to a deeper understanding of the relationship between the Jones polynomial and the representation theory of the fundamental group of the complement of a knot.

In section 2 we recall the basic definitions associated with the A-polynomial. In section 3 we introduce the noncommutative analogues of the spaces and maps used in section 2 and then define the noncommutative A-ideal and the A-basis. In section 4 we derive the orthogonality relationship between the noncommutative A-ideal and the Jones polynomial. In the final section we pose some questions and suggest avenues of future investigation.

2. Characters and Knot Invariants

2.1. $Sl_2\mathbb{C}$**-Characters.** Let $G = < a_i | r_j >$ be a finitely generated group. A representation $\rho : G \to Sl_2\mathbb{C}$ is determined by a choice of matrices A_i in $Sl_2\mathbb{C}$ so that when the relations r_j are rewritten with the a_i replaced with the A_i they are equal to the identity in $Sl_2\mathbb{C}$. In other words, the representations can be identified with the subset of $\prod_{i=1}^n Sl_2\mathbb{C}$, where n is the number of generators of G, that satisfies the equations obtained by requiring the r_j to evaluate to the identity. Denote this subset $\mathrm{Rep}(G)$, and call it the representations of G. Since we only deal with $Sl_2\mathbb{C}$-representations, we suppress references to $Sl_2\mathbb{C}$.

Let $a_{rs}(i)$, $r, s \in \{1, 2\}$, and $i \in \{1, \dots, n\}$ be the function on $\prod_{i=1}^n Sl_2\mathbb{C}$ which yields the entry in the r-th row and s-th column of the i-th entry of each element of $\prod_{i=1}^n Sl_2\mathbb{C}$. The coordinate ring $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ consists of the polynomials in the $a_{rs}(i)$ modulo the ideal generated by

$$a_{11}(i)a_{22}(i) - a_{12}(i)a_{21}(i) - 1,$$

for $i \in \{1, \dots, n\}$. Each relation r_j gives rise to four polynomials corresponding to the four entries of a matrix, and arising from the condition that the relations evaluate to the identity. Let $I(G)$ be the ideal in $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ generated by the polynomials coming from the relations. Finally, $R(G) = C[\prod_{i=1}^n Sl_2\mathbb{C}]/I(G)$ is the *affine representation ring* of G. In more generality, Lubotsky and Magid [LM] proved that the isomorphism class of $R(G)$ is an invariant of the group. The word "affine" refers to the fact that $R(G)$ is the unreduced coordinate ring of the representations. To get the classical representation ring, take the quotient of $R(G)$ by its nilradical $\sqrt{0} = \{p | p^n = 0 \text{ for some } n\}$.

There is a left action of $Sl_2\mathbb{C}$ on $\prod_{i=1}^n Sl_2\mathbb{C}$ by conjugation,

$$A \bullet (A_1, \ldots, A_n) = (AA_1A^{-1}, \ldots, AA_nA^{-1}).$$

This action induces a right action on $C[\prod_{i=1}^n Sl_2\mathbb{C}]$. It is easy to check that the action leaves $I(G)$ invariant, hence the action descends to a right action on $R(G)$. The invariant subring of this action, call it $\chi(G) = R(G)^{Sl_2\mathbb{C}}$, is called the *affine characters* of G. This ring is an invariant of the group G. Once again, to obtain what is classically referred to as the characters, take the quotient of $\chi(G)$ by its nilradical.

For the most frequently encountered groups we are being too careful, the affine representation ring and the affine character ring have trivial nilradical. However, there are examples of groups where the distinction is real [KM].

The affine characters have recently been the subject of scrutiny in works of Bullock, Brumfiel-Hilden, Przytycki-Sikora, and Sikora, [B, BH, PS, S1]. Here is an intrinsic definition due to Sikora. Let $\hat{G}$ be the set of conjugacy classes of G. If W is an element of G we denote the conjugacy class of W by $< W >$. Let $S(\hat{G})$ be the symmetric algebra on $\hat{G}$, that is, polynomials where the variables are conjugacy classes in G and the coefficients are complex numbers. Let J be the ideal generated by all polynomials , $< Id > +2$, and $< AB > + < A^{-1}B >$ $+ < A >< B >$ where A and B range over the elements of G. There is an isomorphism $S(G)/J \to \chi(G)$ induced by sending each $< A >$ into the polynomial corresponding to $-\mathrm{tr}(A)$.

There is a space corresponding to the characters. The action of $Sl_2\mathbb{C}$ on $C[\prod_{i=1}^n Sl_2\mathbb{C}]$ is not good, in the sense that the quotient is not Hausdorff. We sidestep this by defining an equivalence relation on $\mathrm{Rep}(G)$, that yields a Hausdorff space. Let $\rho, \eta \in \mathrm{Rep}(G)$ be equivalent if for every $t \in \chi(G)$, $t(\rho) = t(\eta)$. The resulting quotient space is the character variety of G. An *algebraic subset* of an $\mathbb{C}^n$ is the solution set of a system of polynomial equations. A set of polynomials S *cuts out* an algebraic subset V, if the intersection of the zeroes of elements of S is exactly V. The ideal $I(V)$ of V is the set of all polynomials that vanish on V. It is a consequence of the Nullstellensatz that the ideal of V is the radical of the smallest ideal containing any set of polynomials that cut V out. The coordinate ring of V is the quotient of the ring of polynomials in n variables by $I(V)$. Sometimes it is nice to have a set $X(G)$ corresponding to the characters $\chi(G)$. Of course the set only matters to the extent that its points correspond to maximal ideals in $\chi(G)$. To this end, define a *realization* of $X(G)$ to be an algebraic subset of some $\mathbb{C}^n$, where the coordinates on $\mathbb{C}^n$ correspond to traces of elements of G, whose coordinate ring is isomorphic to $\chi(G)/\sqrt{0}$. The

set $X(G)$ is in one to one correspondence with the character variety of G [CS].

2.2. **Characters of the Torus.** Concentrate on the case of the fundamental group of a torus. Let us think of it as the free Abelian group on λ and μ. The representations correspond exactly to pairs of matrices in $Sl_2\mathbb{C}$ that commute. Next we describe three functions on the space of representations. Let x be the trace of the image of λ, y be the trace of the image of μ and let z be the trace of the image of their product. The affine character ring, $\chi(\pi_1(T^2))$, is generated by x, y, z with relation

$$x^2 + y^2 + z^2 - xyz - 4 = 0.$$

The ring has trivial nilradical. The space $X(\pi_1(T^2))$ is realized as the set of points in $\mathbb{C}^3$ satisfying the equation above. There is a two-fold branched cover of $X(\pi_1(T^2))$ that is used in the definition of knot invariants. Consider $\mathbb{C}^* \times \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the nonzero complex numbers, give it coordinates l and m. In order to see that its coordinate ring is $\mathbb{C}[l, l^{-1}, m, m^{-1}]$, it is helpful to think of $\mathbb{C}^* \times \mathbb{C}^*$ as ordered pairs of diagonal matrices in $Sl_2\mathbb{C}$.

$$\left(\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \right)$$

There is a map $p : \mathbb{C}^* \times \mathbb{C}^* \to X(\pi_1(T^2))$ given by sending each pair of points to its equivalence class. The map p is a two-fold branched cover whose singular points are the four ordered pairs chosen from $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. In terms of our realization of $X(\pi_1(T^2))$, the map sends

$$\left(\begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix}, \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \right)$$

to the triple $(l+l^{-1}, m+m^{-1}, lm+l^{-1}m^{-1})$. Being a two-fold branched cover, p has a deck transformation,

$$\theta : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*,$$

$\theta(l, m) = (l^{-1}, m^{-1})$.

Dual to p, and θ are maps, that we denote $\hat{p}$ and $\hat{\theta}$,

$$\hat{p} : C[X(\pi_1(T^2))] \to \mathbb{C}[l, l^{-1}, m, m^{-1}],$$

and

$$\hat{\theta} : \mathbb{C}[l, l^{-1}, m, m^{-1}] \to \mathbb{C}[l, l^{-1}, m, m^{-1}].$$

One can show that the image of $\hat{p}$ is the fixed subalgebra of $\hat{\theta}$.

Proposition 1. *Suppose that* $V \subset X(\pi_1(T^2))$ *is algebraic and* S *is a set of functions that cuts out* V. *The ideal of* $p^{-1}(V)$ *is the radical of the smallest ideal of* $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ *containing* $\hat{p}(S)$.

Proof. The set $\hat{p}(S)$ cuts out $p^{-1}(V)$, the rest is a standard characterization of the ideal of an algebraic set. □

The coordinate ring $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ can be seen as the ring of fractions of $\mathbb{C}[l, m]$ with respect to the set of monomials in l and m.

Proposition 2. *If* $I \subset \mathbb{C}[l, l^{-1}, m, m^{-1}]$ *is an ideal then its contraction* J *to* $\mathbb{C}[l, m]$ *is* $I \cap \mathbb{C}[l, m]$. *The extension of* J *is just the ideal of* $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ *generated by* J. *Furthermore, the extension of* J *is* I.

□

2.3. **The A-polynomial.** The A-polynomial is defined in [CCGLS]. Let K be a knot in S^3. Let Rep_D denote the set of representations of the fundamental group of the complement of K in which the peripheral subgroup is sent to diagonal matrices. Let $\rho : Rep_D \to \mathbb{C}^* \times \mathbb{C}^*$ be the map that sends each representation to the ordered pair consisting of the upper diagonal entry of the images of the longitude and meridian of the knot. The image of ρ is an algebraic curve. Therefore, its ideal is principal and generated by a Laurent polynomial $B(l, m)$. It can be shown that the image of ρ always has the set $l = 1$ as a component, so $B(l, m) = (l-1)A(l, m)$, where $l-1$ does not divide $A(l, m)$. Properly normalized, the polynomial $A(l, m)$ is an invariant of K, called *the A-polynomial*. It is clear that $A(l, m)$ can be recovered from $B(l, m)$.

There is a map $r : \mathrm{Rep}(\pi_1(S^3 - K))) \to \mathrm{Rep}(\pi_1(T^2))$ given by restriction. This induces a map

$$\hat{r} : \chi(\pi_1(T^2))) \to \chi(\pi_1(S^3 - K)).$$

The kernel of $\hat{r}$ is an ideal $\mathcal{I}(K)$. Notice $\mathcal{I}(K)$ cuts out an algebraic subset V of $X(\pi_1(T^2))$.

Proposition 3. *The radical of the extension of* $\hat{p}(\mathcal{I}(K))$ *in* $\mathbb{C}[l, l^{-1}, m, m^{-1}]$ *is the principal ideal generated by* $B(l, m)$. *That is the ideal* $\mathcal{I}(K)$ *determines the A-polynomial.*

Proof. The ideal $\mathcal{I}(K)$ is exactly the functions that vanish on the image of r. ¿From the definitions, $p^{-1}(\mathrm{im}(r))$ is equal to the image of ρ. The first proposition then finishes the proof. □

You can think of the extension of the ideal $\mathcal{I}(K)$ as the holomorphic sections of a line bundle over $\mathbb{C}^* \times \mathbb{C}^*$. Specifically, the sections of the line bundle associated to the divisor of $\frac{1}{B(l,m)}$.

3. Noncommutative Characters and Knot Invariants

3.1. **The Noncommutative Torus.** The noncommutative torus is a "virtual" geometric space whose algebra of continuous functions is the (irrational) rotation algebra A_θ [Ri]. It is customary to call the algebra of functions itself the noncommutative torus.

The algebra A_θ is usually defined for a real angle of rotation θ, however we consider θ to be any complex number, and let $t = e^{\pi i \theta}$. This algebra can be introduced abstractly by exponentiating the Heisenberg noncommutation relation. That is, A_θ is the closure in a certain C^*-norm of the algebra spanned by l, m, l^{-1}, m^{-1}, subject to the relation $lm = t^2 ml$.

Rieffel [Ri] defined A_θ in a concrete setting. Let $e_{p,q} = t^{-pq} l^p m^q$, where $p, q \in \mathbb{Z}$. He defines his multiplication via the formula

$$e_{p,q} * e_{r,s} = t^{\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right|} e_{p+r,q+s}$$

which, from our approach, is just a consequence of the defining relation. He then considered the closure of this algebra in the norm determined by the left regular representation on $L^2(T^2)$.

For the purpose of this paper we are interested only in the subalgebra of the noncommutative torus consisting of Laurent polynomials in l and m, which we denote by $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$. There is an automorphism

$$\Theta : \mathbb{C}_t[l, l^{-1}, m, m^{-1}] \to \mathbb{C}_t[l, l^{-1}, m, m^{-1}],\ \Theta(e_{p,q}) = e_{-p,-q}$$

Let $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]^\Theta$ be its invariant part (which is spanned by $e_{p,q} + e_{-p,-q}$, $p, q \in \mathbb{Z}$, i.e. by the noncommutative cosines).

In addition, let $\mathbb{C}_t[l, m]$ be the subalgebra of $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ spanned by $e_{p,q}$, with $p, q \geq 0$. This is nothing but the ring of noncommuting polynomials in two variables l and m satisfying the noncommutation relation $lm = t^2 ml$. This ring is frequently referred to as the noncommutative plane.

Proposition 4. *([K], Proposition IV.1.1) The ring $\mathbb{C}_t[l, m]$ is left Noetherian and has no zero divisors.*

We now need to broach the subject of Gröbner bases in $\mathbb{C}_t[l, m]$. As $\mathbb{C}_t[l, m]$ is so close to being commutative, the concepts translate over very easily from the case of two variable polynomials.

We lexicographically order the $l^p m^q$. Hence $l^p m^q < l^r m^s$ if either $p < r$ or $p = r$ and $q < s$. Given $f \in \mathbb{C}_t[l, m]$ we can write $f = \sum \alpha_{p,q} e_{p,q}$ where the sum is finite. The leading term $lt(f)$ of f is the $\alpha_{p,q} l^p m^q$ where the $l^p m^q$ is largest in the lexicographical ordering among those

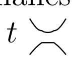
terms with $\alpha_{p,q} \neq 0$. The leading power product is $l^p m^q$ and the leading coefficient is $\alpha_{p,q}$.

Suppose that $u, v, w \in \mathbb{C}_t[l, m]$ and $u = vw$, then we say w divides u on the right and we let $\frac{u}{w} = v$. A *Gröbner basis* for a left ideal I is a collection f_i of elements of the ideal I so that the ideal generated by the leading terms of the f_i is equal to the ideal generated by the leading terms of elements of I. We say the Gröbner basis is *minimal* if no two f_i have the same leading power product. We say the Gröbner basis is *reduced* if no power product in each f_i is divisible by the leading power product of any other f_i.

Proposition 5. *For an ideal of polynomials in $\mathbb{C}_t[l, m]$ there exists a unique minimal, reduced Gröbner basis, consisting of monic polynomials.*

Proof. By changing any statements in [AL] about ideals to statements about left ideals, the proof there goes through verbatim. $\square$

3.2. **The Kauffman Bracket Skein Module.** Let M be an orientable 3-manifold. A framed link in M is an embedding of a disjoint union of annuli into M. In diagrams we will draw only the core of an annulus lying parallel to the plane of the paper (i.e. with blackboard framing).

Two framed links in M are equivalent if there is an isotopy of M taking one to the other. Let $\mathcal{L}$ denote the set of equivalence classes of framed links in M, including the empty link. Fix a complex number t. Consider the vector space, $\mathbb{C}\mathcal{L}$ with basis $\mathcal{L}$. Define $S(M)$ to be the smallest subspace of $\mathbb{C}\mathcal{L}$ containing all expressions of the form  $-t$  $-t^{-1}$ $)($ and $\bigcirc + t^2 + t^{-2}$, where the framed links in each expression are identical outside balls, in which they look like pictured in the diagrams. The Kauffman bracket skein module $K_t(M)$ is the quotient

$$\mathbb{C}\mathcal{L}/S(M).$$

In the case of the cylinder over the torus, $T^2 \times I$, $K_t(T^2 \times I)$ has the structure of an algebra with multiplication given by laying one link over the other. More precisely, to multiply skeins corresponding to links α and β, isotope them so that α lies in $T^2 \times [\frac{1}{2}, 1]$ and β in $T^2 \times [0, \frac{1}{2}]$. Then $\alpha \cdot \beta$ is the element of the skein module represented by the class of the union of these two links in $T^2 \times [0, 1]$. Extend this to a distributive product.

Oriented simple closed curves on the torus up to isotopy are indexed by pairs of relatively prime integers (p, q). Corresponding to (p, q) is a framed link in $T^2 \times I$. Take an annulus in $T^2 \times I$ whose core projects

to a (p,q) curve, so that the annulus runs parallel to the boundary of $T^2 \times I$. As the framed links are unoriented, (p,q) and $(-p,-q)$ give rise to the same link. Denote the link corresponding to the (p,q) by $L_{p,q}$. A standard argument based on the proof that the Kauffman bracket in S^3 is well defined shows that as a vector space, $K_t(T^2 \times I)$ has as basis all links consisting of parallel copies of the $L_{p,q}$.

Let x be $L_{0,1}$, $y = L_{1,0}$ and $z = L_{1,1}$. It is a theorem of Bullock and Przytycki [BP] that $K_t(T^2 \times I)$ is isomorphic to polynomials in three noncommutative variables, x, y and z modulo the ideal generated by

$$t^2x^2 + t^{-2}y^2 + t^2z^2 - txyz - 2(t^2 + t^{-2}),$$

$$txy - t^{-1}yx - (t^2 - t^{-2})z,$$

$$tzx - t^{-1}xz - (t^2 - t^{-2})y,$$

and

$$tyz - t^{-1}zy - (t^2 - t^{-2})x.$$

When $t = -1$ the Kauffman bracket skein module of an arbitrary three-manifold can be made into an algebra. The point is that at $t = -1$ the skein relation allows us to change crossings. To multiply two links, perturb them so that they miss one another and take their union. As crossings don't count, the answer is independent of the perturbation. This extends to make $K_{-1}(M)$ into an algebra for any M. It is a theorem of Bullock [B] that $K_{-1}(M)/\sqrt{0}$ is naturally isomorphic to $\chi(\pi_1(M))/\sqrt{0}$. You can see this isomorphism from the description of $\chi(\pi_1(M))$ due to Sikora given above. The correspondence at $t = -1$ is slightly tricky as the x, y and z given here correspond to $-x$, $-y$ and $-z$ in the relation we gave for the $SL_2\mathbb{C}$-characters of the torus.

In [FG] the following theorem is proved.

Theorem 1. *There exists an isomorphism of algebras*

$$\hat{p} : K_t(T^2 \times I) \to \mathbb{C}_t[l, l^{-1}, m, m^{-1}]^{\Theta}$$

determined by

$$\hat{p}(L_{p,q}) = e_{(p,q)} + e_{(-p,-q)},\ p, q \in \mathbb{Z}.$$

3.3. **The noncommutative A-basis.** Let M be an oriented homology 3-sphere. Let K be a knot in M. Let X be the complement of a regular neighborhood of K. The space X is a compact manifold with boundary homeomorphic to T^2. The meridian of the knot and longitude are well defined up to sign, by requiring that the meridian bound a disk in the regular neighborhood and the longitude bound a Seifert surface, and their intersection number in the boundary of X is 1. There is a map $\hat{r} : K_t(T^2 \times I) \to K_t(X)$ obtained by gluing the cylinder over the torus into the complement of the knot at the $T^2 \times \{0\}$ end so that the meridian goes to the meridian and the longitude goes to the longitude. Let $\mathcal{I}_t(K)$ be the left ideal which is the kernel of $\hat{r}$. This ideal is called the peripheral ideal of the knot K. The map $\hat{p}$ takes $\mathcal{I}_t(K)$ to $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$. Let $\mathcal{J}_t(K)$ be the extension of $\mathcal{I}_t(K)$ to a left ideal in $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$. Finally, the noncommutative A-ideal $\mathcal{A}(K)$ is the contraction of $\mathcal{J}_t(K)$ to $\mathbb{C}_t[l, m]$.

The *A-basis* of a knot K at the complex number t is the reduced minimal Gröbner basis of the left ideal $\mathcal{A}_t(K)$.

Recall for any 3-manifold $K_{-1}(M)/\sqrt{0}$ is isomorphic to $\chi(\pi_1(M))/\sqrt{0}$. For a cylinder over a torus the radical of both rings is trivial, hence

$$K_{-1}(T^2 \times I) = \chi(\pi_1(T^2)).$$

It can be shown that under this isomorphism, the ideal $\mathcal{I}_{-1}(K)$ cuts out the image of the characters of $\pi_1(S^3 - K)$, and hence its extension cuts out the variety used to determine the A-polynomial.

Theorem 2. *The noncommutative A-basis at $t = -1$ determines the A-polynomial*

□

Let us denote by $\mathcal{I}_t$ the left ideal that is the kernel of the epimorphism

$$\hat{\rho} : K_t(T^2 \times [0, 1]) \to K_t(D^2 \times T).$$

In [FG] it is proved that $\mathcal{I}_t$ is generated by $L_{(0,1)} + t^2 + t^{-2}$ and $L_{(1,1)} + t^{-3}L_{(1,0)}$. To compute the noncommutative A-basis one first shows that the contraction of $\mathcal{I}_t$ is generated by $m^2 + (t^2 + t^{-2})m + 1$ and $t^{-3}l^2m^2 + t^{-5}l^2m + t^{-1}m + t$. These are just the contractions of the generators of $\mathcal{I}_t$ to the noncommutative plane. Then an easy application of Buchberger's Algorithm yields the minimal reduced Gröbner basis $\{m^2 + (t^2 + t^{-2})m + 1, l^2m + t^{-2}l^2 - m - t^2\}$. To get this to correspond to the value of the A-polynomial there are two subtleties. The first is that as we set it up, the meridian bounds instead of the longitude. The second is the negative sign arising in the correspondence between skeins and characters. If you exchange l with $-m$ and m with

$-l$, then the least common multiple of the two polynomials will be $l-1$ which is as one would expect.

4. Relation with the Jones polynomial

Assume in this section that t is not a root of unity. The definition and normalization of the Jones-Wenzl idempotents will be used as in [Li]. Let S_c be the skein in the solid torus obtained by plugging the c-th Jones-Wenzl idempotent into the core of the solid torus. The Kauffman bracket skein module of the solid torus has the set $\{S_c\}$ as a basis. Let $\hat{K}_t(S^1 \times D^2)$ be the vector space of formal sums $\sum_c z_c S_c$ where the z_c are complex numbers and the c range over the natural numbers starting at 0.

The double of the solid torus is $S^1 \times S^2$. Any skein α in $S^1 \times S^2$ can be represented by a linear combination of framed links that miss $1 \times S^2$. Hence, any skein in $S^1 \times S^2$ can be represented as a skein in a punctured ball, and it has a Kauffman bracket. It follows that $K_t(S^1 \times S^2)$ is canonically isomorphic to $\mathbb{C}$. There is a pairing between skeins in $K_t(S^1 \times D^2)$. If $\alpha, \beta \in K_t(S^1 \times D^2)$ are represented by a single framed link each, take the union of two copies of the solid torus, identified along their boundaries, with the link representing α in one and the link representing β in the other. As this yields a skein in $S^1 \times S^2$ we get a complex number by taking the Kauffman bracket as above. This can be extended bilinearly to give a pairing,

$$K_t(S^1 \times D^2) \otimes \hat{K}_t(S^1 \times D^2) \to \mathbb{C},$$

for although the sum is infinite only finitely many terms are nonzero. In this way we identify $\hat{K}_t(S^1 \times D^2)$ with $K_t(S^1 \times D^2)^*$.

There is a representation of the Kauffman bracket skein algebra of the cylinder over the torus into endomorphisms of $\hat{K}_t(S^1 \times D^2)$. Glue the cylinder onto $S^1 \times D^2$ along the 0-end of the cylinder so that longitudes and meridians go to longitudes and meridians. The matrix of a skein in the cylinder over the torus as a matrix with respect to the basis S_c, is of bounded width. That is, there is an integer n so that if $|i-j| > n$ the ij-entry of the matrix is zero. This can be seen, as the matrix induced is the adjoint of the matrix corresponding to the endomorphism of $K_t(S^1 \times D^2)$ induced by gluing the 1-end of the cylinder to $S^1 \times D^2$. Notice that if $\hat{Z} \in \hat{K}_t(S^1 \times D^2)$ then the annihilator of $\hat{Z}$ in the Kauffman bracket skein module of the cylinder over the torus is a left ideal.

Let $K \subset S^3$ be a framed knot. Let X be the complement of an open regular neighborhood of the knot. There is a pairing,

$$K_t(S^1 \times D^2) \otimes K_t(X) \to \mathbb{C},$$

obtained by gluing the solid torus into the knot so that the meridian of the solid torus goes to the meridian of the knot and the blackboard longitude goes to the framing of the knot. To pair two skeins, take their union and then take the Kauffman bracket in S^3 of the result. By using the empty skein in X we get a linear functional,

$$Z(K) : K_t(S^1 \times D^2) \to \mathbb{C}.$$

Let $\kappa(K, c)$ be the value of $Z(K)$ on S_c. We can then represent $Z(K)$ by

$$\hat{Z}(K) = \sum_c \kappa(K, c)\sigma_c$$

where $\{\sigma_c\}$ is the basis dual to $\{S_c\}$. It is worth noting that the $\kappa(K, c)$ are the colored Kauffman brackets of the knot (which are a version of the colored Jones polynomials of the knot). Indeed, the c-th coefficient of the series expansion is computed by plugging S_c along the core of the regular neighborhood of the knot and evaluating in S^3, that is by coloring the knot with the c-th Jones-Wenzl idempotent and evaluating the result in the skein space of the plane.

Let $\mathcal{F}_t(K)$ be the annihilator of $\hat{Z}(K)$ in $K_t(T^2 \times I)$. This is a left ideal that we call the *formal ideal* of K.

Theorem 3. *The ideal $\mathcal{I}_t(K)$ lies in $\mathcal{F}_t(K)$.*

Proof. Recall that a skein α in $T^2 \times I$ is in $\mathcal{I}_t(K)$ if when you glue the 0 end of $T^2 \times I$ to X, the skein is equivalent to 0 in $K_t(X)$. Let $\alpha \in \mathcal{I}_t(K)$ and $x \in K_t(S^1 \times K^2)$. Then $\alpha Z(K)$ is a functional on $K_t(S^1 \times D^2)$ and its value on x is computed by embedding the skein a in the knot complement by gluing the cylinder over the torus to the knot complement, and then gluing the solid torus with the skein x in it to the knot complement. But the skein α can be transformed into the zero skein by skein moves taking place entirely in the complement of x, hence the value of the functional is zero. Thus the functional itself is zero. □

If α is a skein in $K_t(T^2 \times I)$, then α has a matrix representation coming from the action on the skein module of the solid torus, with basis $\{S_c\}$. The theorem shows that the rows of this matrix are orthogonal to the vector $\hat{Z}(K)$.

5. The view beyond

The first question about this set up is whether $\mathcal{I}_t(K) = \mathcal{F}_t(K)$. Even in restricted cases, like when the colored Jones polynomials of the knot are all trivial this is a provocative question. This gives the first glimmer of how one might go about proving that the Jones polynomial distinguishes the unknot.

The fact that the peripheral ideal annihilates the vector whose coordinates are the colored Kauffman brackets of the knot gives rise to a family of relations for the generators which depend on the parameter t. Do these depend smoothly on t? If yes, they can be differentiated and evaluated at $t = -1$ to yield derivatives of the A-polynomial of a knot. How are those to be understood in terms of the $Sl_2\mathbb{C}$-representation of the knot? The pairing used in the orthogonality relation is clearly some sort of averaging process. Are there integral formulas for the colored Jones polynomials, where the integral is taken over the character variety of the knot?

What kind of information about incompressible surfaces in a knot complement can be extracted from the noncommutative invariant. Can you see the placement of incompressible surfaces in the knot complement from the colored Jones polynomials of the knot?

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